1 Complex Numbers

“The beginning is the most important part of the work”
- Plato

1.1 Why Complex Numbers?

A complex number consists of a real part and an imaginary part and is generally represented as:

$$A = x + iy \qquad [1\text{-}1]$$

where $i = \sqrt{-1}$.

One can also represent the complex number pictorially in a Argand diagram as:

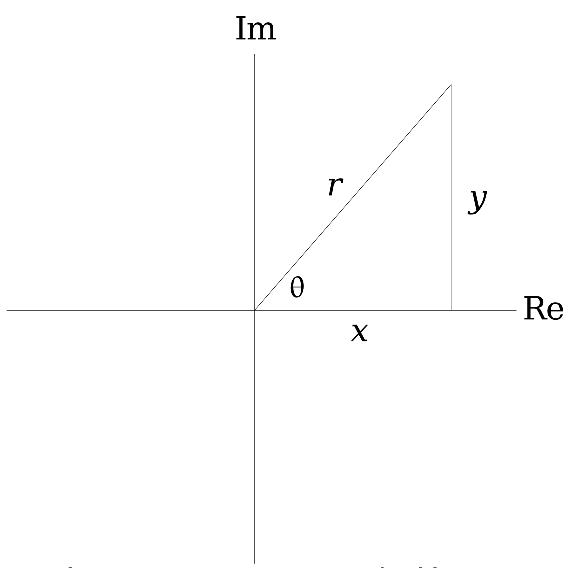

Figure 1-1: Argand diagram for the graphical representation of complex numbers

where Im and Re indicate the imaginary and real axes.

The *complex conjugate* of A is A* and is defined as:

$$A^* = x - iy \qquad [1\text{-}2]$$

Where do complex numbers come from? Under what circumstances would anyone want to contemplate using them? Essentially, complex numbers arise from a consideration of the roots of certain polynomials. A very simple example would be the roots of $x^2 + 4 = 0$:

$$x^2 = -4$$
$$x = \pm\sqrt{-4}$$
$$= \pm 2\sqrt{-1}$$

In this case the solution of the equation is an *imaginary number* but in general a complex number of the form introduced above in equation 1-1 can be expected. Since $\sqrt{-1}$ arises so frequently it is given the symbol i in the sciences (the symbol j is used extensively in engineering as i is used for electric current there).

It is interesting to briefly consider the history of numbers. The first humans had no need for numbers and would therefore probably not have understood the concept. Ultimately, of course, numbers were invented for use in counting .. taking stock of animals and items of trade. The first set of numbers was undoubtedly simply a set of integers beginning at 1 and increasing from there .. we're all familiar with them as we learn them at a very early age. This is very obvious and intuitive to us; anyone that counts uses these numbers. It turned out however that this number system had to be altered to include zero. Why? Well, basically, in order to count nothing you need a number for it. Originally used in India for practical calculations it is now an integral part of our number systems. So, the simple number system was altered out of necessity to produce what we now refer to as the natural numbers.

Negative numbers were introduced to represent debts, again in India. Yet again the number system was altered to include negative numbers into what we now refer to as the integers. There were those in ancient Greece who considered negative solutions to equations to be false and 'absurd' so the use of negative numbers has not been without its controversies.

Next, in order to express fractional quantities yet a new set of numbers was needed. Using fractions of integers we come up with the *rational numbers*. Thus:

$$\frac{1}{2} = 0.5$$
$$\frac{1}{3} = 0.333\ldots$$

but we still do not have a complete set of numbers. For example, the square root of 2 cannot be expressed as a rational number. That is, it cannot be expressed as a ratio of integers. It is referred to as an *irrational number.* So again the number set was added to with rational and irrational numbers. Together they make up the set of real numbers.

As mentioned above, the solutions to some equations require the use of the root of a negative number. Of course this idea would have driven the ancient Greeks around the bend! How can one take the root of a negative number? Nonetheless it is necessary to use this concept to solve these equations so we again alter our number set to include the root of -1 in the form of complex numbers. Each time that a new type of number was needed in the past it was added to our set of numbers.

This still doesn't really explain complex numbers, however. If you look at the general solution of polynomials you find that for a polynomial of degree n there are n factors from which we can extract n roots of the equation[1]. For example:

$$x^3 - 3x^2 + 6x - 4 = 0$$

This polynomial is of order three and has the three roots:

$$x = 1, x = 1 + \sqrt{-3}, x = 1 - \sqrt{-3}$$

and in general for a polynomial:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + N = 0$$

there are *n* factors:

$$(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)\ldots(x - \nu) = 0$$

such that:
A = the sum of $-\alpha$, $-\beta$, $-\gamma$ … $-\nu$
B = the sum of all their products two at at time
C = the sum of all their products three at a time
.
.
N = the product of all taken together

You can see from the above example equation that there are two complex roots for the equation. Furthermore, they are conjugates (see below) of each other. Thus, the use of x + *i*y to represent these types of numbers.

1.2 Properties of Complex Numbers

1 There is a fascinating and very readable translation of an article published by Leonhard Euler in 1751 on complex numbers and their origin. Please see the references list.

The conjugate of a complex number differs from the complex number in that the opposite sign is used in front of the term containing *i*.

The inverse of *i* equals -*i*:

$$\frac{1}{i}=\frac{1}{i}\cdot\frac{i}{i}=\frac{i}{-1}=-i \qquad [1\text{-}3]$$

The sums and products of conjugate complex numbers are real:

$$\begin{gathered}(x+iy)+(x-iy)=2\text{x}\\(x+iy)\cdot(x-iy)=x^2+y^2\end{gathered} \qquad [1\text{-}4a]$$

and the difference between complex conjugates is imaginary:

$$(x+iy)-(x-iy)=2\text{iy} \qquad [1\text{-}4b]$$

From the Argand diagram in figure 1-1, the length of the line **r** is called the *modulus* of complex number A:

$$r=\sqrt{x^2+y^2}=|A| \qquad [1\text{-}5]$$

Some properties of the modulus:

$$\begin{gathered}|A|=|A^*|\\|A||A^*|=|A|^2\\|A_1A_2|=|A_1||A_2|\end{gathered} \qquad [1\text{-}6]$$

We can extract the real and imaginary parts of the complex number using the appropriate functions:

$$\begin{gathered}z=x+iy\\\text{Re}(z)=x\\\text{Im}(z)=y\end{gathered} \qquad [1\text{-}7]$$

One can also see an angle, θ , in the Argand diagram, figure 1-1. Thus, the complex number can also be represented in polar coordinate form:

$$A=re^{i\theta} \qquad [1\text{-}8]$$

Euler's theorem (see Appendix II) is stated as:

$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

so that:

$$A = r\,e^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$$

From the Argand diagram (figure 1-1):

$$\cos(\theta) = \frac{x}{r}, \sin(\theta) = \frac{y}{r}$$
$$\theta = \arccos(\frac{x}{r}) = \cos^{-1}(\frac{x}{r})$$
$$= \arcsin(\frac{y}{r}) = \sin^{-1}(\frac{y}{r})$$

(Note that $\sin^{-1}$ and $\cos^{-1}$ do not refer to the inverses of these functions!). So:

$$A = r[\frac{x}{r} + i\frac{y}{r}]$$
$$= x + iy$$

hence the intimate association of the complex number with a phase angle, θ .

The product of two complex numbers can be done in two ways; either using the formal complex notation definition of the number (equation 1-1) or using the polar coordinate form (equation 1-7). Thus, for two complex numbers:

$$A_1 = 4 + i5 \qquad A_2 = 3 + i$$

or in polar notation:

$$A_1 = 6.40\,e^{0.8960\mathrm{i}}\,A_2 = 3.16\,e^{0.3218\mathrm{i}}$$

the product of A_1 and A_2 using the complex notation is:

$$A_1 \cdot A_2 = (4 + i5)(3 + i)$$
$$= 12 + 4\mathrm{i} + 15\mathrm{i} + 5\mathrm{i}^2$$
$$= 7 + 19\mathrm{i}$$

or in polar notation:

$$A_1 \cdot A_2 = 6.40\,e^{0.8960\,i} \cdot 3.16\,e^{0.3218\mathrm{i}}$$
$$= 20.25\,e^{1.2178\mathrm{i}}$$

(the phase angles here are in radians). We can show that these two results are equivalent by converting one into the other. Let's convert the complex notation answer into polar notation:

$$r = \sqrt{x^2 + y^2} = \sqrt{7^2 + 19^2} = 20.25$$

$$\theta = \sin^{-1}\left(\frac{y}{r}\right) = \sin^{-1}\left(\frac{19}{20.25}\right) = 1.2178$$

$$r = 20.25 e^{1.2178i}$$

The product of a complex number and its conjugate is real and positive:

$$A \cdot A^* = r e^{(i\phi)} r e^{(-i\phi)} = r^2 = x^2 + y^2 = \left\| A^2 \right\| \quad [1\text{-}9]$$

The inverse of a complex number is:

$$\frac{1}{x + iy}$$

however, this is not in standard form (equation [1-1]) so:

$$\frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \quad [1\text{-}10]$$

Thus, we can divide two complex numbers:

$$A = x_a + iy_a \quad B = x_b + iy_b$$
$$\frac{A}{B} = \frac{x_a + iy_a}{x_b + iy_b} = (x_a + iy_a)\left[\frac{x_b}{x_b^2 + y_b^2} - i\frac{y_b}{x_b^2 + y_b^2}\right]$$
$$= \frac{x_a x_b + y_a y_b}{x_b^2 + y_b^2} + i\frac{x_b y_a - x_a y_b}{x_b^2 + y_b^2} \quad [1\text{-}11]$$

We can, of course, also do division using the polar form:

$$A = r_A e^{i\theta_A} \quad B = r_B e^{i\theta_B}$$
$$\frac{A}{B} = \frac{r_A}{r_B} e^{i(\theta_A - \theta_B)}$$

As with the product of two complex numbers in polar form, this is equivalent to the standard form quotient:

$$\frac{A}{B}=\frac{r_A}{r_B}e^{i(\theta_A-\theta_B)}$$
$$=\frac{r_A}{r_B}(\cos(\theta_A-\theta_B)+isin(\theta_A-\theta_B))$$
$$=\frac{r_A}{r_B}[(\cos(\theta_A)\cos(\theta_B)+\sin(\theta_A)\sin(\theta_B))+i(\sin(\theta_A)\cos(\theta_B)-\sin(\theta_B)\cos(\theta_A))]$$
$$=\frac{r_A}{r_B}[(\frac{x_a x_b+y_a y_b}{r_A r_B})+i(\frac{y_a x_b-y_b x_a}{r_A r_B})]$$
$$=\frac{1}{r_B^2}[(x_a x_b+y_a y_b)+i(y_a x_b-y_b x_a)]$$
$$=\frac{x_a x_b+y_a y_b}{x_b^2+y_b^2}+i\frac{y_a x_b-y_b x_a}{x_b^2+y_b^2}$$

which is of the same form as [1-11].

From the Euler theorem, sine and cosine can be expressed in terms of exponentials:

$$\cos(\phi)=\frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\sin(\phi)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

c_numbers??

1.3 Problems

1.4 References

1. D.W. Jordan and P. Smith, *Mathematical Techniques,* Oxford University Press, 1994.
2. L. Euler, *Recherches sur les racines imaginaires des equations,* translated by Todd Doucet. Available from the online Euler Archive at http://math.dartmouth.edu/~euler/.